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CHEMOTAXIS–FLUID COUPLED MODEL FOR SWIMMING BACTERIA WITH NONLINEAR DIFFUSION: GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR

Marco Di Francesco

Department of Pure and Applied Mathematics
Via Vetoio, Loc. Coppito
67100 L'Aquila, Italy

Alexander Lorz and Peter Markowich

Department of Applied Mathematics and Theoretical Physics
University of Cambridge, Wilberforce Road
Cambridge CB3 0WA, UK

Abstract. We study the system

$$\begin{cases} c_t + u\cdot\nabla c = \Delta c - nf(c) \\ n_t + u\cdot\nabla n = \Delta n^m - \nabla\cdot(n\chi(c)\nabla c) \\ u_t + u\cdot\nabla u + \nabla P - \eta\Delta u + n\nabla\phi = 0 \\ \nabla\cdot u = 0. \end{cases}$$

arising in the modelling of the motion of swimming bacteria under the effect of diffusion, oxygen-taxis and transport through an incompressible fluid. The novelty with respect to previous papers in the literature lies in the presence of nonlinear porous–medium–like diffusion in the equation for the density n of the bacteria, motivated by a finite size effect. We prove that, under the constraint $m \in (3/2, 2]$ for the adiabatic exponent, such system features global in time solutions in two space dimensions for large data. Moreover, in the case $m = 2$ we prove that solutions converge to constant states in the large–time limit. The proofs rely on standard energy methods and on a basic entropy estimate which cannot be achieved in the case $m = 1$. The case $m = 2$ is very special as we can provide a Lyapounov functional. We generalize our results to the three–dimensional case and obtain a smaller range of exponents $m \in (m^*, 2]$ with $m^* > 3/2$, due to the use of classical Sobolev inequalities.

1. **Introduction.** A vital characteristic of living organisms is the ability to sense signals in the environment and adapt their movements accordingly. This behaviour enables them to locate certain chemical substances (e. g. nutrients), avoid predators or find animals of the same species. A quite general model for such kind of systems is

$$\begin{cases} \partial_t c + \nabla\cdot J_c = g(n,c) \\ \partial_t n + \nabla\cdot J_n = 0 \end{cases} \tag{1.1}$$

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as a first approximation. Here, c is the concentration of a chemical, n is a cell density, J_n is the cell flux, J_c the chemical flux and $g(n,c)$ is the production / consumption rate of the chemical. When the presence of the living organism activates the production of a certain chemical substance, it is called chemotaxis. The typical example for chemotaxis is the amoebae *Dictyostelium*: when they are running out of nutrients, they emit a chemical, cyclic Adenosine Monophosphate (cAMP), which attracts other amobae and together they form some kind of transition to a multicellular organism. This fruiting body can survive food shortage. In many other situations, the chemical substance is consumed by the living organism and the chemotaxis effect is replaced by a transport towards a nutrient. A typical example of that is the *Bacillus subtilis* commonly found in soil: when they are swimming in water, such organisms are subject to a drift up to the gradient of the concentration of oxygen (oxygentaxis). These two phenomena (chemotactical transport vs. transport towards a nutrient) give rise to essentially different situations from the mathematical viewpoint. In both cases, the existence theory in suitable functional spaces and the asymptotic behavior can present several difficulties.

The simplest and most classical example in the framework of chemotactical movements is the Patlak-Keller-Segel model [14, 15] where $J_n := n\chi\nabla c - D_n\nabla n$, $J_c := D_c\nabla c$ and $g(n,c) := \alpha n - \beta c$. So the cells perform a biased random walk in the direction of the chemical gradient and the chemical diffuses, it is produced by the cells and it degrades.

$$\begin{cases} \partial_t c = D_c\Delta c + \alpha n - \beta c \\ \partial_t n + \nabla\cdot(n\chi\nabla c - D_n\nabla n) = 0 \end{cases} \tag{1.2}$$

Nondimensionalizing and assuming fast diffusion and slow degradation of the chemical, gives:

$$\begin{cases} -\Delta c = n \\ \partial_t n + \nabla\cdot(\chi n\nabla c - \nabla n) = 0. \end{cases} \tag{1.3}$$

In this paper we shall rather be concerned with the case of *consumption of the nutrient* by the bacteria under the simultaneous effect of the transport by surrounding water. In the framework provided by (1.1), one typically has

$$J_n := n\chi(c)\nabla c + un - D_n\nabla n.$$

Here u is the velocity field of the incompressible fluid (water). So the bacteria diffuse, swim upwards oxygen gradients and they are transported with the fluid.

Moreover, the oxygen also diffuses, is also transported by the fluid and it is consumed proportional to the density of cells n and a cut-off function $f(c)$, which models an inactivity threshold of the bacteria due to low oxygen supply: therefore we have

$$\begin{aligned} J_c &:= -D_c\nabla c + uc \\ g(n,c) &:= -nf(c). \end{aligned}$$

A standard non–dimensionalization leads to the simplified system

$$\begin{cases} c_t + u\cdot\nabla c = \Delta c - nf(c) \\ n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(n\chi(c)\nabla c) \end{cases} \tag{1.4}$$

Obviously an equation for the fluid has to be added. We take the velocity field u of the fluid to be described by an incompressible Navier-Stokes-type equation with pressure P and viscosity η and model the gravitational force by $n\nabla\phi$. These gives the following set of equations:

$$\begin{cases} c_t + u \cdot \nabla c = \Delta c - nf(c) \\ n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n\chi(c)\nabla c) \\ u_t + u \cdot \nabla u + \nabla P - \eta \Delta u + n\nabla\phi = 0 \\ \nabla \cdot u = 0 \end{cases} \tag{1.5}$$

In [21] the authors proposed these model equations and performed experiments showing large-scale convection patterns.

The experimental set-up corresponds to mixed-type boundary conditions. For simplicity here we use no-flux conditions for c and n and zero Dirichlet for u. Moreover, since the fluid flow is slow, we can use the Stokes equation instead of the Navier-Stokes system. So the system looks like

$$\begin{cases} c_t + u \cdot \nabla c = \Delta c - nf(c) \\ n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n\chi(c)\nabla c) \\ u_t + \nabla P - \eta \Delta u + n\nabla\phi = 0 \\ \nabla \cdot u = 0. \end{cases} \tag{1.6}$$

We remark that the sign in front of the n-term in the equation on c is different from the one used in the Keller-Segel model. For the systems (1.5) and (1.6) there is a local existence result [20]. Moreover, in [11] the authors proved global existence for (1.6) with weak potential or small initial c. To our knowledge, these are the only results on (1.5) and (1.6). However, attention has recently been focused on coupled kinetic–fluid systems firstly introduced in [6] which have a similar mathematical flavor; also refer to [12, 9] about the studies of the Vlasov-Fokker-Planck equation coupled with the compressible or incompressible Navier-Stokes or Stokes equations, where the main tool used to prove the global existence of weak solutions or hydrodynamic limits is an existing entropy inequality.

For the Navier-Stokes and Stokes equations see [17, 19] and references therein for the detailed mathematical theory.

A key open mathematical problem is then to establish global existence of solutions for general (large) data for (1.6). Although related to a different phenomenon, the Keller–Segel model for chemotaxis offers a good paradigm on the possible singular behaviour of solutions. The problem of global existence vs. blow–up has been completely solved for the elliptic-parabolic Keller–Segel model (1.3) in $\mathbb{R}^2$. [3] summarizes the results, *i.e.* there is a critical mass M, below M we have global existence and above M we have finite-time blow-up. For the parabolic-parabolic Keller–Segel model (1.2) recent progress has been achieved in [8]. For more references on the general Keller-Segel system, the interested reader can refer to recent work [2, 3, 8]. Kinetic models for chemotaxis can be found in [10].

Several authors in the chemotaxis literature have recently addressed the prevention of finite–time blow-up (*overcrowding*, from the modelling viewpoint) by assuming e.g. that, due to the finite size of the bacteria, the random mobility increases for large densities. This leads to a nonlinear porous–medium–like diffusion instead of a linear one, see e. g. [7, 16]. Other ways to model prevention of overcrowding can be found in [13, 5, 4]. The result, e.g. in case of the elliptic–parabolic Keller–Segel system, is that solutions exist globally in time no matter how large the initial mass is.

In this paper we modify our system (1.6) in a similar fashion. More precisely, we address the global existence for large data for the full chemotaxis–fluid coupled system (1.6) with nonlinear diffusion for n instead of a linear one. More precisely, we shall study the model system

$$\begin{cases} c_t + u \cdot \nabla c = \Delta c - n f(c) \\ n_t + u \cdot \nabla n = \Delta n^m - \nabla \cdot (n \chi(c) \nabla c) \\ u_t + \nabla P - \eta \Delta u + n \nabla \phi = 0 \\ \nabla \cdot u = 0. \end{cases} \tag{1.7}$$

In the two dimensional case we shall prove that, when the adiabatic exponent m ranges in the interval $(3/2, 2]$ (therefore we are in the *slow* diffusion range), the system (1.7) admits a global-in-time solutions for general (large) initial data. The same result holds in dimension three in a smaller range of exponents $m \in (m^*, 2]$. To perform this task, we make use of energy (or entropy) a priori estimates for n and c which cannot be obtained in the case $m = 1$. The basic estimate is contained in section 3.3. A short discussion about the range of exponents m for which our results are valid is contained in the Remark 1. In the case $m = 2$ we can provide a monotone decreasing (Lyapounov) functional for the system (1.7) (involving the three variables n, c and u); as a byproduct of that, we shall be able to prove asymptotic long–time convergence towards constant states. In future work we shall explore the possibility of employing the entropy specific for the porous medium equation in order to derive rates of convergence as t tends to ∞.

The paper is structured as follows. In section 2 we state the problem in detail and state our results on global existence and asymptotic behaviour for large times. In section 3 we prove the global existence of solutions for large data. In section 4 we address the asymptotic behariour for large times. In section 5 we propose a model system alternative to (1.7), for which we briefly explain how to construct global-in-time solutions in a similar fashion as for (1.7). Finally, in section 6 we prove the result in the three–dimensional case.

2. **Preliminaries and results.**

We shall study the model system

$$\begin{cases} c_t + u \cdot \nabla c = \Delta c - n f(c) \\ n_t + u \cdot \nabla n = \Delta n^m - \nabla \cdot (n \chi(c) \nabla c) \\ u_t + \nabla P - \eta \Delta u + n \nabla \phi = 0 \\ \nabla \cdot u = 0 \end{cases} \tag{2.1}$$

posed on a bounded domain $\Omega \subset \mathbb{R}^2$ subject to the boundary conditions

$$\begin{aligned} &\partial_\nu n^m(x,t) = 0 \\ &\partial_\nu c(x,t) = 0 \qquad\qquad x \in \partial\Omega,\ t \geq 0 \\ &u(x,t) = 0 \end{aligned}$$

and to the initial data

$$n(x,0) = n_0(x) \geq 0, \quad c(x,0) = c_0(x) \geq 0, \quad u(x,0) = u_0(x) \quad x \in \Omega.$$

In a special case, namely when $m = 2$, we shall be able to extend our results to the whole space $\Omega = \mathbb{R}^2$. Next, we require

$$\begin{aligned}
&f : [0, +\infty) \to [0, +\infty) \text{ is a } C^1 \text{ function with } f(0) = 0,\ f'(c) > 0\\
&\chi : [0, +\infty) \to [0, +\infty) \text{ is continuous}\\
&\nabla\phi \in L^\infty(\Omega) \text{ and } \phi \text{ independent of time}\\
&\frac{3}{2} < m \le 2
\end{aligned} \tag{2.2}$$

We remark that these assumptions imply positivity of f.

Definition 2.1 (weak solution)**.** A quadruple (c, n, u, P) is said to be a weak solution of (1.7) if

$$\begin{aligned}
&-\int_0^T\int_\Omega \psi_t n + \int_\Omega \psi n_0 - \int_\Omega \nabla\psi \cdot un = \int_0^T\int_\Omega \Delta\psi n^m + \int_0^T\int_\Omega \nabla\psi\cdot(n\chi(c)\nabla c)\\
&-\int_0^T\int_\Omega \psi_t c + \int_\Omega \psi c_0 - \int_\Omega \nabla\psi \cdot uc = \int_0^T\int_\Omega c\Delta\psi + \int_0^T\int_\Omega nf(c)\psi\\
&-\int_0^T\int_\Omega \tilde\psi_t \cdot u + \int_\Omega \tilde\psi u_0 = \int_0^T\int_\Omega u\cdot\Delta\tilde\psi + \int_0^T\int_\Omega n\nabla\phi\cdot\tilde\psi
\end{aligned}$$

holds for all $\psi \in C^\infty([0,T]\times\Omega)$ and all $\tilde\psi \in C^\infty([0,T]\times\Omega)$ with value in either $\mathbb{R}^2$ or $\mathbb{R}^3$ and $\nabla\cdot\tilde\psi = 0$.

Our aim is to prove global existence of weak solutions of the system (2.1) as stated in the two following theorems.

Theorem 2.2 (quadratic case, full space)**.** *Suppose that $m = 2$ and $\Omega = \mathbb{R}^2$; suppose that the set of assumptions (2.2) is satisfied and*

$$\begin{aligned}
&n_0^m \in H^1(\mathbb{R}^2), (1+|x|)n_0 \in L^1(\mathbb{R}^2),\\
&c_0 \in H^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2), u_0 \in H^1(\mathbb{R}^2).
\end{aligned}$$

Then for **all** *$T > 0$ there exists a weak solution (c, n, u, P) with*

$$c \in L^\infty(0,T; W^{1,p}(\mathbb{R}^2)) \cap L^2(0,T; H^2(\mathbb{R}^2)) \quad \textit{for all } p \in [1, +\infty)$$

$$n\ln(n) \in L^\infty(0,T; L^1(\mathbb{R}^2)), n \in L^2(0,T; H^1(\mathbb{R}^2))$$

$$u \in L^2(0,T; H^2(\mathbb{R}^2))$$

Theorem 2.3 (bounded domain)**.** *Suppose that $3/2 < m \le 2$, Ω is a bounded domain with smooth boundary and*

$$\begin{aligned}
&n_0^m \in H^1(\Omega),\\
&c_0 \in H^1(\Omega)\cap L^\infty(\Omega), u_0 \in H^1(\Omega).
\end{aligned}$$

Then for **all** *$T > 0$ there exists a weak solution (c, n, u, P) with*

$$c \in L^\infty(0,T; W^{1,p}(\Omega)) \cap L^2(0,T; H^2(\Omega)) \quad \textit{for all } p \in [1, +\infty)$$

$$n \in L^\infty(0,T; L^{3-m}(\Omega)) \cap L^2(0,T; H^1(\Omega))$$

$$u \in L^2(0,T; H^2(\Omega))$$

Remark 1. The authors consider this set of results as a first step, a major scope being to solve the dichotomy between the possible existence of a critical exponent m^* (such that blow up occurs as m gets smaller than m^*) and the possible global existence for large data for all m (even for $m = 1$). The motivation behind the two–dimensional range $(3/2, 2)$ is (so far) purely technical. On the other hand, we remark that the exponent $m = 2$ has often turned out to be crucial in the study of the regularity of weak solutions to the porous medium equation, cf. [22], as the gradient of the solutions becomes unbounded near the free boundary for $m > 2$. The lower bound comes from using a Gagliardo-Nirenberg inequality for n, we refer to section 3.5 for details.

The proof of the above two theorems is based on uniform estimates on a regularized system. More precisely, in order to solve the equation for n we regularize the chemotaxis term $\nabla \cdot (n\chi(c)\nabla c)$ in the evolution equation for n in (2.1) and we replace the nonlinear *slow* diffusion term with a non-degenerate one. Then, the equation for c is easily solvable as convection–diffusion–absorption equation and the equation for u can be solved via standard Stokes theory. We then perform a set of uniform estimates on the regularized solution needed to pass to the limit on all time intervals $[0, T]$. The strategy we use in the estimates are crucially based on a free–energy inequality, which is not achievable in the case $m = 1$. Also, remark 1 explains the restrictions on m in more detail.

As a byproduct of the global existence of solutions, we shall investigate the long time behaviour. Our result holds in the case $m = 2$ *in any space dimension* and it reads as follows.

Theorem 2.4 (Asymptotic behaviour)**.** *Suppose that $f(c) > 0$ for $c > 0$, $m = 2$, Ω bounded and all assumptions of Theorem 2.3 are satisfied. Then $n(t)$ converges to $\int n_0/|\Omega|$ in $L^1(\Omega)$, $c(t)$ converges to 0 in $L^2(\Omega)$ and $u(t)$ converges to 0 in $L^2(\Omega)$ for $t \to \infty$.*

The global existence result will be proven to hold as well in three space dimensions provided one restricts the range of adiabatic exponents to $m \in (m^*, 2]$ for a certain $m^* > 3/2$, cf. section 6.

Although the model system (1.6) has been used for numerical computations in the biophysical literature [21], we remark that it could be more realistic to include both the effect of gravity (potential force) on cells and the effect of the chemotactic force on fluid, which leads to the more complicated model system

$$\begin{cases} c_t + u \cdot \nabla c = \Delta c - nf(c) \\ n_t + u \cdot \nabla n = \Delta n^m - \nabla \cdot (n\chi(c)\nabla c) + \nabla(n\nabla\phi) \\ u_t + \nabla P - \eta\Delta u + n[\nabla\phi - \chi(c)\nabla c] = 0 \\ \nabla \cdot u = 0 \end{cases} \tag{2.3}$$

Although the coupling here is even stronger and more nonlinear, the same results as in Theorems 2.2 and 2.3 holds. We shall discuss this in detail in section 5.

3. **Global existence.**

This section is devoted to the proof of the Theorems 2.2 and 2.3. Therefore, throughout the whole section we shall assume $x \in \mathbb{R}^2$.

3.1. **Approximation.**

We shall work on the approximating system

$$\begin{cases} c_t^\epsilon + u^\epsilon \cdot \nabla c^\epsilon = \Delta c^\epsilon - n^\epsilon f(c^\epsilon) \\ n_t^\epsilon + u^\epsilon \cdot \nabla n^\epsilon = m\nabla \cdot \left((|n^\epsilon| + \epsilon)^{m-1} \nabla n^\epsilon \right) - \nabla \cdot \left(n^\epsilon \chi(c^\epsilon) \frac{\nabla c^\epsilon}{1+\epsilon|\nabla c^\epsilon|} \right) \\ u_t^\epsilon + \nabla p^\epsilon - \eta \Delta u^\epsilon + n^\epsilon \nabla \phi = 0 \\ \nabla \cdot u^\epsilon = 0 \end{cases} \tag{3.1}$$

where $\epsilon > 0$ is a small parameter. The global existence for system (3.1) can be easily proven by freezing the velocity field u in the equations for c and n and by solving them globally in time using standard results in [18]. The estimates on n and c are independent of u because of the incompressibility condition on u. Then, one solves the equation for u by following the standard theory in [17]. A simple bootstrap argument can be easily closed to provide a strong solution in the limit. As some of the needed estimates will be reproduced in the sequel, we shall omit the details of the above mentioned procedure.

To increase readability we suppress the supercript ϵ in the following till section 3.8, although we work on the approximating system.

In order to prove the nonnegativity property $n(x,t) \geq 0$ for all $t > 0$, we define $n_- := max(-n, 0)$, multiply equation $(3.1)_2$ by $(n_-)^{p-1}$, integrate over Ω and use the estimate

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int (n_-)^p + (p-1)m\epsilon^{m-1}\int n_-^{p-2}|\nabla n_-|^2 \\ &\qquad \leq \max|\chi(c)|(p-1)\|\nabla c\|_\infty \left(\int_\Omega (n_-)^p\right)^{1/2} \left(\int_\Omega (n_-)^{p-2}|\nabla n_-|^2\right)^{1/2} \\ &\qquad \leq \frac{p-1}{2} m\epsilon^{m-1}\int n_-^{p-2}|\nabla n_-|^2 + C\int (n_-)^p. \end{aligned}$$

Then the Gronwall inequality easily implies $n_- = 0$ for all times since $n_0 \geq 0$.

In order to prove $c(x,t) \geq 0$ we write the equation for c as

$$c_t + u \cdot \nabla c = \Delta c - ncf'(\eta)$$

with η between 0 and c. We then obtain $c(x,t) \geq 0$ by testing against the evolution of the negative part c_- as for n.

The conservation of the total mass $\int_\Omega n(x,t)dx = \int_\Omega n_0(x)dx := M$ follows easily from the boundary conditions for c, n and u on a bounded domain. On the whole space this can be performed by using a suitable cut-off in a standard way (cf. [22]).

In the next subsections we shall address the global existence of weak solutions to (2.1). We shall perform estimates on the approximating system which imply compactness of the sequence $(c^\epsilon, n^\epsilon, u^\epsilon, P^\epsilon)$ and its consistency in the limit as $\epsilon \to 0$. In the following we shall assume enough regularity in order to have all the estimates well justified. In order to make this argument rigorous, one should regularize the initial data and prove that the approximated system (3.1) provides a smooth enough solution if the initial data are, say, C^∞ and then pass to the limit in the mollifying parameter. It can be easily seen (e.g. via Green's function representation) that such an argument does not produce any technical difficulty in our case, therefore we shall omit it.

3.2. **Global boundedness of c.** As a first step, we prove that c has uniformly bounded L^p norms, $p \in [1, +\infty]$, for all times if c is initially bounded. Integration by parts and the condition $\nabla \cdot u = 0$ imply

$$\frac{d}{dt}\int_\Omega c^p = -p(p-1)\int_\Omega c^{p-2}|\nabla c|^2 dx - p\int_\Omega c^{p-1} f(c) n dx \leq 0$$

for all $p \in [1, +\infty)$, which yields

$$\|c(t)\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)} =: c_M \qquad \text{for all } t \geq 0. \tag{3.2}$$

3.3. **Free energy.** We now compute the evolution of the *free energy* of the density n. Here we distinguish between two cases, namely $m = 2$ and $3/2 < m < 2$.

3.3.1. *Quadratic case.* We start by considering the case $m = 2$. We compute the evolution of the logarithmic entropy $\int n \log n$ as follows:

$$\begin{aligned}\frac{d}{dt}\int_\Omega n\ln(n) &= -\int_\Omega \frac{1}{n}\nabla n\cdot\big(-un + \nabla(n+\epsilon)^2 - n\chi(c)(1+\epsilon|\nabla c|)^{-1}\nabla c\big)\\ &\leq -2\int_\Omega \frac{n+\epsilon}{n}|\nabla n|^2 + \int_\Omega \chi(c)|\nabla n||\nabla c| \leq -\|\nabla n\|_{L^2}^2 + \frac{1}{4}\max_{0\leq c\leq c_M}|\chi(c)|^2\|\nabla c\|_{L^2}^2,\end{aligned} \tag{3.3}$$

where we have used $\nabla\cdot u = 0$. The above estimate has to be combined together with the following one

$$\frac{1}{2}\frac{d}{dt}\int_\Omega c^2 = \int_\Omega c\Delta c - \int_\Omega f(c)nc = -\|\nabla c\|_{L^2}^2 - \int_\Omega f(c)nc. \tag{3.4}$$

Let us set $K := \max\limits_{0\leq c\leq c_M}|\chi(c)|^2/2$. Combining (3.3) and (3.4) yields

$$\frac{d}{dt}\int_\Omega n\ln(n) + K\frac{d}{dt}\int_\Omega c^2 \leq -\|\nabla n\|_{L^2}^2 - K/2\|\nabla c\|_{L^2}^2 - K\int_\Omega f(c)nc. \tag{3.5}$$

To conclude the above estimate we distinguish between the two cases of Ω being a bounded domain or the full space $\mathbb{R}^2$.

In case of a bounded domain we simply integrate w.r.t. to t in (3.5) to get

$$\begin{aligned}&\int_\Omega n\ln(n)(t) + K\int_\Omega c^2(t) + \int_0^t \|\nabla n(s)\|_{L^2}^2 ds\\ &+ K/2\int_0^t \|\nabla c(s)\|_{L^2}^2 ds + \int_0^t K\int_\Omega f(c)nc(s)ds \leq \int_\Omega n_0\ln(n_0) + K\int_\Omega c_0^2\end{aligned} \tag{3.6}$$

and since the function $n \mapsto n\log(n)$ is bounded from below, the estimate (3.6) can be used to obtain uniform bounds on ∇n and ∇c in $L^2([0,+\infty]\times\Omega)$.

In case of $\Omega = \mathbb{R}^2$, we have to control the behavior of n as $|x| \to +\infty$ similarly to [11]. To perform this task, we multiply $(3.1)_2$ by the smooth function $\xi = \sqrt{1+x^2}$ and integrate to get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2}\xi n dx &= \int_{\mathbb{R}^2} nu\cdot\nabla\xi dx + \int_{\mathbb{R}^2} n^2\Delta\xi dx + 2\int_{\mathbb{R}^2}\epsilon n\Delta\xi dx\\ &+ \int_{\mathbb{R}^2}\chi(c)n(1+\epsilon|\nabla c|)^{-1}\nabla c\cdot\nabla\xi dx \leq \|n\|_1\|u\|_\infty\|\nabla\xi\|_\infty + C\|\nabla n\|_2\|n\|_1\|\Delta\xi\|_\infty\\ &\qquad + 2\epsilon\|n\|_1\|\Delta\xi\|_\infty + \max_{0\leq c\leq c_M}|\chi(c)|\|n\|_2\|\nabla c\|_2\|\nabla\xi\|_\infty\\ &\qquad\leq \delta\|u\|_\infty^2 + \delta\|\nabla n\|_2^2 + \delta\|\nabla c\|_2^2 + C(\delta,\epsilon)\end{aligned} \tag{3.7}$$

We used that $\|\nabla\xi\|_\infty$ and $\|\Delta\xi\|_\infty$ are bounded and the Nash-inequality $\|n\|_2^2 \le C\|n\|_1\|\nabla n\|_2$. Moreover we have

$$\begin{aligned}\int_{\mathbb{R}^2} n\ln\left(\frac{1}{n}\right)\mathbb{1}_{n\le 1}dx &\le \int_{\mathbb{R}^2} n\ln\left(\frac{1}{n}\right)\mathbb{1}_{e^{-\xi}\le n}dx + \int_{\mathbb{R}^2} n\ln\left(\frac{1}{n}\right)\mathbb{1}_{n\le e^{-\xi}}dx \\ &\le \int_{\mathbb{R}^2}\xi n dx + C\int_{\mathbb{R}^2} n^{1/2}\mathbb{1}_{n\le e^{-\xi}}dx \\ &\le C + \int_{\mathbb{R}^2}\xi n\,dx. \end{aligned}\tag{3.8}$$

From (3.5) we have

$$\int_{\mathbb{R}^2} n(t)\ln(n(t))\mathbb{1}_{n\ge 1} + K\int c^2(t)\,dx \le \\ -\int_0^t \|\nabla n\|_{L^2}^2 - \int_0^t K/2\|\nabla c\|_{L^2}^2 + \int_{\mathbb{R}^2} n(t)\ln\frac{1}{n(t)}\mathbb{1}_{n<1}dx.$$

From (3.7) and (3.8), we obtain by choosing δ small enough

$$\int_{\mathbb{R}^2} n(t)\ln(n(t))\mathbb{1}_{n\ge 1} + K\int c^2(t)\,dx \le -\frac{1}{2}\int_0^t \|\nabla n\|_{L^2}^2 - \int_0^t K/4\|\nabla c\|_{L^2}^2 + C(\epsilon)t. \tag{3.9}$$

3.3.2. *Subquadratic case.* In the case $m \in (3/2, 2)$ we have to estimate the functional $\int n^{m-1}$ in a similar way as above. Here the final estimate only holds in a bounded domain.

$$\begin{aligned}\frac{d}{dt}\int_\Omega n^{m-1} &= (m-1)\int_\Omega n^{m-2}\nabla\cdot\big(-un + \nabla(n+\epsilon)^m - n\chi(c)(1+\epsilon|\nabla c|)^{-1}\nabla c\big) \\ &\ge (m-1)(2-m)\left(m\int_\Omega n^{m-3}(n+\epsilon)^{m-1}|\nabla n|^2 - \int_\Omega n^{m-2}\chi(c)|\nabla n||\nabla c|\right) \\ &\ge \frac{m(2-m)}{m-1}\int_\Omega |\nabla n^{m-1}|^2 - (2-m)\int_\Omega \chi(c)|\nabla n^{m-1}||\nabla c|. \end{aligned}\tag{3.10}$$

Therefore, integrating with respect to time we obtain

$$\frac{m(2-m)}{2(m-1)}\int_0^t\int |\nabla n^{m-1}(x,s)|^2 dxds \le \\ \frac{(2-m)(m-1)}{2m}\max_{0\le c\le c_M}|\chi(c)|^2\int_0^t\int |\nabla c(x,s)|^2 dxds \\ + \int n^{m-1}(x,t)dx - \int n_0(x)^{m-1}dx \tag{3.11}$$

which combined with

$$\int n^{m-1}(x,t)dx = \int_{n\le 1} n^{m-1}(x,t)dx + \int_{n\ge 1} n^{m-1}(x,t)dx \le |\Omega| + M,$$

($M := \int n dx$) and with (3.4) (as in the quadratic case) gives

$$\int_0^t\int |\nabla n^{m-1}(x,s)|^2 dxds \le C, \tag{3.12}$$

for a constant C depending only on the initial data.

3.4. $L^2_{x,t}$**–estimate of** n**.** Having at hand a space–time L^2–estimate for n will turn out to be very useful in the sequel. We show here how to obtain it very simply starting from the estimates derived in subsection 3.3. In the quadratic case $m = 2$, a direct use of Poincaré inequality yields

$$\int_0^t \|n(s)\|^2_{L^2} ds \le C \int_0^t (\|\nabla n(s)\|^2_{L^2} + 1) ds$$

and the term at the right hand side is uniformly bounded w.r.t. t due to (3.9). In the case $m \in (3/2, 2)$, we use Gagliardo–Nirenberg interpolation inequality as follows

$$\int_0^t \|n(s)\|^2_{L^2} ds = \int_0^t \|n(s)^{m-1}\|^{\frac{2}{m-1}}_{L^{\frac{2}{m-1}}} ds \le C \int_0^t (\|\nabla n^{m-1}\|^{\frac{1}{(m-1)}}_{L^2} \|n^{m-1}\|^{\frac{1}{(m-1)}}_{L^{\frac{1}{m-1}}} + 1)\, ds$$

and the conservation of the total mass of n (together with the condition $m > 3/2$) easily implies

$$\int_0^t \|n(s)\|^2_{L^2} ds \le C \int_0^t \left[\|\nabla n^{m-1}(s)\|^2_{L^2} + 1\right] ds \le C(1 + t) \tag{3.13}$$

in view of the estimates in section 3.3. Moreover, the right hand side is bounded because of (3.12).

3.5. **Improved estimate of** ∇c**.** Now we recall the following estimate, as a consequence of [17, Theorem 6, page 100]

$$\begin{aligned} \int_0^t \|u(s)\|^2_{L^\infty} ds \le& C \int_0^t \|u_t(s)\|^2_{L^2} + \|u(s)\|^2_{H^2} ds \\ \le& C \int_0^t \left[\|\nabla \phi\|_{L^\infty} \|n(s)\|^2_{L^2}\right] ds + \|u_0\|^2_{H^1}, \end{aligned} \tag{3.14}$$

where C depends on the initial data and the right hand side is bounded because of (3.13). From the estimate (3.14) and from the equation for u we easily deduce $u \in C([0, T]; H^1(\Omega))$ and therefore we have $u \in L^{\bar{r}}((0, T) \times \Omega)$ for all $\bar{r} < \infty$. By using integration by parts we obtain

$$\begin{aligned} \frac{d}{dt} \|\nabla c\|^p_{L^p} &\le p(p-2) \int |\nabla c|^{p-2} |D^2 c| |c_t| dx + p \int |\nabla c|^{p-2} |\Delta c| |c_t| dx \\ &\le C_p \|\nabla c\|^{p-2}_{L^p} \|D^2 c\|_{L^p} \|c_t\|_{L^p} \le C_p \left(\|\nabla c\|^p_{L^p} + \|D^2 c\|^p_{L^p} + \|c_t\|^p_{L^p}\right) \end{aligned}$$

and therefore, on an arbitrary time interval $[0, T]$,

$$\begin{aligned} \|\nabla c\|^p_{L^\infty_t L^p_x} \le& C \left(1 + \|c_t\|^p_{L^p_t L^p_x} + \|\nabla c\|^p_{L^p_t L^p_x} + \|D^2 c\|^p_{L^p_t L^p_x}\right) \\ \le& C \|n f(c)\|^p_{L^p_t L^p_x} + C \end{aligned} \tag{3.15}$$

where the last inequality is justified by regularity results in [18, Chapter IV] due to the above stated regularity for u. The constant C may depend on T but it is well defined for all $T > 0$. Gagliardo–Nirenberg interpolation and the conservation of the mass for n imply, for $\beta \le 1$,

$$\|n\|_p = \|n^\beta\|^{1/\beta}_{p/\beta} \le C(\|\nabla n^\beta\|^{\bar{\alpha}/\beta} + \|n\|_1)$$

with $\bar\alpha = (\beta - \beta/p)(\beta)^{-1} = 1 - 1/p$. In order to control the $L^p((0,T)\times\Omega)$-norm of n, we need $\bar\alpha p/\beta = 2$. So $p = 2\beta + 1$. We start by using (3.12)

$$\int_0^t \int |\nabla n^{m-1}(x,s)|^2 dxds \le C,$$

By choosing $\beta = \beta_0 := m-1$, we have $\|n\|_{L_t^{p_0} L_x^{p_0}} < \infty$ for

$$p_0 := 2\beta_0 + 1 = 2m - 1 > 2 \tag{3.16}$$

and from (3.15)

$$\|\nabla c\|_{L_t^\infty L_x^{p_0}} \le C + C\|n\|_{L_t^{p_0} L_x^{p_0}} \le C \tag{3.17}$$

We remark here that the condition $m > 3/2$ is crucial in order to make the space integrability exponent p_0 larger than 2.

3.6. **Improved estimate of ∇n for $m < 2$.** We continue by estimating the L^α-norm of n as follows:

$$\begin{aligned}\frac{d}{dt}\int n^\alpha dx \le & -\frac{4\alpha m(\alpha-1)}{(\alpha+m-1)^2}\int |\nabla n^{\frac{\alpha+m-1}{2}}|^2 dx \\ & + \alpha(\alpha-1)\int n^{\alpha-1}\nabla n\cdot(1+\epsilon|\nabla c|)^{-1}\nabla c\, dx.\end{aligned}$$

Let us define $I := \alpha(\alpha-1)\int n^{\alpha-1}\nabla n\cdot(1+\epsilon|\nabla c|)^{-1}\nabla c\, dx$ and estimate

$$|I| \le C\int n^{\frac{\alpha+1-m}{2}}|\nabla n^{\frac{\alpha+m-1}{2}}||\nabla c|\, dx \le C\|n^{\frac{\alpha+1-m}{2}}\|_{L^q}\|\nabla n^{\frac{\alpha+m-1}{2}}\|_{L^2}\|\nabla c\|_{L^p}$$

with $\frac12 = \frac1p + \frac1q$, $p, q > 2$. Choosing $\alpha = 3-m$ and using $\|n^{2-m}\|_q \le (\|\nabla n\|_2 + 1)^{2-m}$ implies

$$|I| \le C\|n^{2-m}\|_q\left(\int |\nabla n|^2 dx\right)^{\frac12}\|\nabla c\|_{L^p} \le C\|\nabla n\|_2(\|\nabla n\|_2+1)^{2-m}\|\nabla c\|_{L^p} \tag{3.18}$$

So we obtain

$$\int_0^t\int |\nabla n(x,s)|^2 dxds \le C, \tag{3.19}$$

3.7. **Estimate for the H^{-1}-norm of n_t.** To estimate the H^{-1}-norm of n_t we solve

$$\begin{cases} n_t = \Delta\psi \\ \left.\frac{\partial\psi}{\partial\nu}\right|_{\partial\Omega} = 0. \end{cases} \tag{3.20}$$

Multiplying the equation for n by ψ and integrating by parts we obtain

$$\begin{aligned}
&\int_\Omega \psi n_t - \int_\Omega \nabla\psi\cdot un = \int_\Omega \Delta\psi(n+\epsilon)^m + \int_\Omega \nabla\psi\cdot(n\chi(c)\nabla c)\\
&-\int_\Omega|\nabla\psi|^2 - \int_\Omega\nabla\psi\cdot un = 1/(m+1)\frac{d}{dt}\int(n+\epsilon)^{m+1} + \int_\Omega\nabla\psi\cdot(n\chi(c)\nabla c)\\
&1/(m+1)\frac{d}{dt}\int(n+\epsilon)^{m+1} + 1/2\int_\Omega|\nabla\psi|^2\\
&\qquad\le \|u\|^2_{2(m+1)/(m-1)}\|n\|^2_{m+1} + C\|n\|^2_{m+1}\|\nabla c\|^2_{2(m+1)/(m-1)}\\
&1/(m+1)\frac{d}{dt}\int(n+\epsilon)^{m+1} + 1/2\int_\Omega|\nabla\psi|^2\\
&\qquad\le \|\nabla u\|_2^{m+1}\|n\|^{m+1}_{m+1} + C\|n\|^{m+1}_{m+1}\|\nabla c\|^{m+1}_{2(m+1)/(m-1)} + C.
\end{aligned}$$

Now, a simple interpolation argument shows that (3.17) and

$$\int_0^t \|\Delta c(s)\|_{L^2}^2 ds \leq C$$

(which is a consequence of (3.15) for $p = 2$) imply that

$$\|\nabla c\|_{m+1}^{2(m+1)/(m-1)}$$

is bounded on any time interval $[0, T]$. Therefore, by Gronwall inequality we obtain that $\int_0^t |\nabla\psi|^2$ is bounded.

Next we show that $\|n_t\|_{H^{-1}} \leq \|\nabla\psi\|^2$. We observe that $\|n_t\|_{H^{-1}} = \|\nabla w\|_2$ where

$$\begin{cases} n_t = \Delta w \\ w|_{\partial\Omega} = 0. \end{cases} \tag{3.21}$$

Hence, we have

$$\begin{aligned} \|n_t\|_{H^{-1}}^2 &= \sup_{f\in H_0^1, \|f\|_{H_0^1}\leq 1} \langle n_t, f\rangle = \sup_{f\in H_0^1, \|f\|_{H_0^1}\leq 1} \int \nabla w \cdot \nabla f \\ &= \sup \int -\Delta w f = \sup \int -\Delta\psi f = \sup \int \nabla\psi\cdot\nabla f \leq \|\nabla\psi\|_2^2. \end{aligned}$$

We have therefore proven that n_t is uniformly bounded in $L^2([0, T], H^{-1}(\Omega))$ on any time interval $[0, T]$.

3.8. **Passing to the limit.** So now we solve our system for $\epsilon \to 0$ and apply

Lemma 3.1 (Aubin-Lions compactness lemma)**.** *Let X_0, X_1 and X_2 be three Banach spaces with $X_0 \subset X_1 \subset X_2$. Suppose that X_0 is compactly embedded in X_1 and that X_1 is continuously embedded in X_2; suppose also that X_0 and X_2 are reflexive spaces. For $1 < p, q < \infty$ let*

$$W := \{u \in L^p([0, T]; X_0) | \dot{u} \in L^q([0, T]; X_2)\}.$$

Then the embedding of W into $L^p([0, T]; X_1)$ is also compact.

with $X_0 := H^1$, $X_1 := L^2$ and $X_2 := H^{-1}$. We recall

$$H^1(\Omega) \overset{\text{compact}}{\hookrightarrow} L^2(\Omega) \overset{\text{continuous}}{\hookrightarrow} H^{-1}(\Omega).$$

Since n_t^ϵ is bounded in $L^2(0, T; H^{-1}(\Omega))$ and n^ϵ is bounded in $L^2(0, T; H^1(\Omega))$, there is a subsequence n^ϵ that converges in $L^2(0, T; L^2(\Omega))$ to n. Since c_t^ϵ is bounded in $L^2(0, T; L^2(\Omega))$ and c^ϵ is bounded in $L^2(0, T; H^2(\Omega))$, there is a subsequence c^ϵ that converges in $L^2(0, T; H^1(\Omega))$ to c. Similarly, for u: Since u_t^ϵ is bounded in $L^2(0, T; L^2(\Omega))$ and u^ϵ is bounded in $L^2(0, T; H^1(\Omega))$ ($\|u_t^\epsilon\|_2 + \|u^\epsilon\|_{L^2(0,T;H^1(\Omega))} \leq C\|n\|_2$), there is a subsequence u^ϵ that converges in $L^2(0, T; L^2(\Omega))$ to u.

Given a test function ψ

$$\begin{aligned} &\int_0^T \int_\Omega \psi n_t + \int_0^T \int_\Omega \psi u \cdot \nabla n = \int_0^T \int_\Omega \psi \Delta n^m - \int_0^T \int_\Omega \psi \nabla \cdot (n\chi(c)\nabla c) \\ &- \int_0^T \int_\Omega \psi_t n + \int_\Omega \psi n_0 - \int_\Omega \nabla\psi \cdot un = \int_0^T \int_\Omega \Delta\psi n^m + \int_0^T \int_\Omega \nabla\psi \cdot (n\chi(c)\nabla c) \end{aligned}$$

we obtain the desired weak solution in the limit in the sense of Definition 2.1. In particular, the strong convergence of n^ϵ in $L^2(0, T; L^2(\Omega))$ and of c^ϵ in $L^2(0, T; H^1(\Omega))$ is sufficient to pass to the limit in the term $n\chi(c)\nabla c$.

3.9. **Improved regularity.** To obtain the full regularity stated in Theorems 2.2 and 2.3, a procedure quite similar to [18, Chapter IV] is required: splitting the domain using a partition of unity and solving the equation in coordinate patches using a kernel. Since this is very technical, we skip the proof.

4. **Asymptotic behaviour.** This section is devoted to the proof of Theorem 2.4. We shall therefore tackle the asymptotic behavior for large times for the system

$$\begin{cases} c_t + u \cdot \nabla c = \Delta c - n f(c) \\ n_t + u \cdot \nabla n = \Delta n^m - \nabla \cdot (n\chi(c)\nabla c) \\ u_t + \nabla P - \eta \Delta u + n\nabla\phi = 0 \\ \nabla \cdot u = 0. \end{cases} \tag{4.1}$$

We remark that such result holds in any space dimension. For this section only, we assume $f(c) > 0$ for $c > 0$ and Ω bounded. Moreover we shall restrict ourselves to the case $m = 2$. This is due to the fact that we can provide a Lyapounov functional for the system (4.1) involving the three variables n, c and u at the same time only if $m = 2$.

As for the boundary conditions, we shall always work on a bounded domain Ω with zero Neumann boundary conditions for c and n and with zero Dirichlet data for u

From the u-equation we obtain

$$\begin{aligned} \frac{d}{dt}\|u\|_2^2 + 2\|\nabla u\|_2^2 = 2\int_\Omega nu\nabla\phi = -2\int_\Omega \nabla n \cdot u\phi \\ \leq 2C\|\nabla n\|_2\|u\|_2 \leq C\|\nabla n\|_2^2 + \|\nabla u\|_2^2. \end{aligned}$$

So together with the free energy estimate it follows that

$$\frac{d}{dt}\int_\Omega \left[n\ln(n) + Kc^2 + \lambda|u|^2\right] \leq -1/2\|\nabla n\|_2^2 - K/2\|\nabla c\|_2^2 - \lambda\|\nabla u\|_2^2 - K\int_\Omega f(c)nc.$$

Together with the entropy dissipation, we obtain

$$\int_0^\infty \int_\Omega \left[|\nabla n|^2 + |\nabla c|^2 + |\nabla u|^2 + f(c)nc\right] dx < \infty.$$

As the above time integral converges on $[0, +\infty)$, there exists a diverging sequence t_k such that $\int_\Omega |\nabla n(x,t_k)|^2$, $\int_\Omega |\nabla c(x,t_k)|^2$ and $\int_\Omega (f(c)nc)(x,t_k)$ tend to 0 as $k \to +\infty$. In particular, $\int_\Omega |\nabla n(x,t_k)|^2$ is bounded. So, we can extract a subsequence $n(t_k)$ such that $n(t_k)$ converges strongly in $L^2(\Omega)$ to $\bar{n}$ and $\nabla n(t_k)$ converges weakly in $L^2(\Omega)$ to $\nabla \bar{n} = 0$. Therefore, $\bar{n}$ is constant and by mass conservation it is equal to $\int n_0/|\Omega|$. Similarly, we have that $\bar{c}$ and $\bar{u}$ are constant. Since we have the term $\int_\Omega f(c)nc$ in the dissipation, it follows that $\bar{c} = 0$. The boundary conditions force $\bar{u} = 0$.

Now, as $n(t_k)$ is uniformly bounded in L^2, we have, for R large enough (here n denotes $n(t_k)$)

$$\begin{aligned}
&\left|\int (n\log n - \bar n \log \bar n)\,dx\right| \le \int_{n<R} |n\log n - \bar n\log\bar n|dx + \int_{n\ge R} |n\log n - \bar n\log\bar n|dx \\
&\le \int_{n<R} |n\log n - \bar n\log\bar n|dx + \int_{n\ge R} \frac{|n\log n - \bar n\log\bar n|}{C+Cn^2}(C+Cn^2)dx \\
&\le \int_{n<R} |n\log n - \bar n\log\bar n|dx + \frac{C}{\sqrt{R}}\int_{n\ge R}(C+Cn^2)dx \\
&\le \int_{n<R} |n\log n - \bar n\log\bar n|dx + \frac{C}{\sqrt{R}}
\end{aligned}$$

and the first term in the last line above is converging to zero as $k \to +\infty$ in view of Lebesgue's dominated convergence theorem. As the above computation holds for any R large enough, we have proven that $\int n(t_k)\log n(t_k) \to \int \bar n \log \bar n$ and the monotonicity of the Lyapounov functional implies that the whole family $\int n(t)\log n(t)dx$ converges to $\int \bar n\log\bar n$. A trivial variant of the so called Csiszar–Kullback inequality [1] implies then

$$\|n_k - \bar n\|_1^2 \le C\int_\Omega [n_k \ln(n_k) - \bar n \ln(\bar n)].$$

Therefore, $n(t) \to \bar n$ in $L^1(\Omega)$, $c(t) \to 0$ in $L^2(\Omega)$ and $u(t) \to 0$ in $L^2(\Omega)$ for $k \to \infty$, where the last two assertions are also consequences of the monotonicity of the Lyapounov functional.

5. **Self-consistent model.**

Let us recall the self-consistent model:

$$\begin{cases} c_t + u\cdot\nabla c = \Delta c - nf(c) \\ n_t + u\cdot\nabla n = \Delta n^m - \nabla\cdot(n\chi(c)\nabla c) + \nabla(n\nabla\phi) \\ u_t + \nabla P - \eta\Delta u + n[\nabla\phi - \chi(c)\nabla c] = 0 \\ \nabla\cdot u = 0. \end{cases} \tag{5.1}$$

Multiplying the u-equation by u and integrating gives

$$\frac{d}{dt}\|u\|_2^2 + \eta\|\nabla u\|_2^2 = -\int_\Omega n\nabla\phi\cdot u - \int_\Omega n\chi(c)\nabla c\cdot u.$$

Defining $F(c)$ as the primitve of $\chi(c)$, we have for $3/2 < m < 2$

$$\begin{aligned}
\int_\Omega n\chi(c)\nabla c\cdot u &= \int_\Omega n\nabla F(c)\cdot u = \int_\Omega \nabla n F(c)\cdot u = \int_\Omega \frac{1}{m-1} n^{2-m}\nabla n^{m-1}F(c)\cdot u \\
&= C(m)\max_{0\le c\le c_m}(|F(c)|)\|n^{2-m}\|_{1/(2-m)}\|\nabla n^{m-1}\|_2\|u\|_{1/(m-1.5)} \\
&\le C\|\nabla n^{m-1}\|_2\|\nabla u\|_2.
\end{aligned} \tag{5.2}$$

(5.2) holds also when $m = 2$. Therefore it follows that

$$\frac{d}{dt}\|u\|_2^2 + \eta/2\|\nabla u\|_2^2 \le C\|n\|_2^2 + C(m)\|\nabla n^{m-1}\|_2^2 \le C(\|\nabla n^{m-1}\|_2^2 + 1). \tag{5.3}$$

So $u \in L^4((0,T)\times\Omega)$. This is enough regularity on u to obtain the results of subsections 3.5 and 3.6. For the subsection 3.9, we need $u \in L^\infty((0,T)\times\Omega)$, to this end we show that $n\nabla\phi$, $n\nabla c$ are in $L^q((0,T)\times\Omega)$ for $2<q<p:=p_0$, this also means we need $m>3/2$. With $1/q=1/r+1/p$, we have

$$\begin{aligned}\int_0^T \|n\nabla c\|_q^q &\le C\|\nabla c\|^q_{L^p_xL^\infty_t}\int_0^T \|n\|_r^q \\ &\le C\|\nabla c\|^q_{L^p_xL^\infty_t}\int_0^T \left(\|\nabla n\|_2^{1-1/r}\|n\|_1^{1/r}+\|n\|_1\right)^q .\end{aligned}$$

Since $\int_0^T \|\nabla n\|_2^{(1-1/r)q}<\infty$ for q close enough to 2, we obtain from

$$\|n\nabla c\|_{L^q_xL^q_t}\le C\|\nabla c\|_{L^p_xL^\infty_t}\left(\|\|\nabla n\|_2^{(1-1/r)}\|_{L^q_t}+C\right)$$

that $\|n\nabla c\|_{L^q_xL^q_t}<\infty$.

6. The three–dimensional case.

Theorem 6.1. *1. Suppose that $m=2$, $\Omega=\mathbb{R}^3$, the set of assumptions (2.2) is satisfied and*

$$\begin{aligned}&n_0\in H^1(\Omega),(1+|x|)n_0\in L^1(\Omega),\\ &c_0\in H^1(\Omega)\cap L^\infty(\Omega),u_0\in H^1(\Omega).\end{aligned}$$

Then for **all** *$T>0$ there exists a weak solution (c,n,u,P) with*

$$\begin{gathered}c\in L^\infty(0,T;W^{1,p}(\Omega))\cap L^2(0,T;H^2(\Omega))\quad \text{for all } p\in[1,+\infty)\\ n\ln(n)\in L^\infty(0,T;L^1(\Omega)),n\in L^2(0,T;H^1(\Omega))\\ u\in L^2(0,T;H^2(\Omega))\end{gathered}$$

2. Suppose that $m^*:=\frac{7+\sqrt{217}}{12}\le m\le 2$, $\Omega\subset\mathbb{R}^3$ bounded domain with smooth boundary and

$$\begin{aligned}&n_0\in H^1(\Omega),\\ &c_0\in H^1(\Omega)\cap L^\infty(\Omega),u_0\in H^1(\Omega).\end{aligned}$$

Then for **all** *$T>0$ there exists a weak solution (c,n,u,P) with*

$$\begin{gathered}c\in L^\infty(0,T;W^{1,p}(\Omega))\cap L^2(0,T;H^2(\Omega))\quad \text{for all } p\in[1,+\infty)\\ n\in L^\infty(0,T;L^{3-m}(\Omega))\cap L^2(0,T;H^1(\Omega))\\ u\in L^2(0,T;H^2(\Omega))\end{gathered}$$

full space. We just have to check the moment control in section 3.3.1: There we replace the inequality $\|n\|_2^2\le C\|n\|_1\|\nabla n\|_2$ in $\mathbb{R}^2$ by $\|n\|_2\le C\|n\|_1^{2/5}\|\nabla n\|_2^{3/5}$ in $\mathbb{R}^3$.

bounded domain. We need to check the Gagliardo-Nirenberg interpolation, in

- section 3.5: $p=\frac{1}{3}(5m-2)$ instead of (3.16) (this gives $m>8/5$ since $p>2$ in section 3.6),
- section 3.6: the Sobolev imbedding used in inequality (3.18) still works but with the restriction
$$(2-m)q\le 6, \tag{6.1}$$
- section 3.7: Gagliardo-Nirenberg interpolation to estimate $\|\nabla c\|^{m+1}_{2(m+1)/(m-1)}$ and require its integrability, gives the condition $0\le 6m^2-7m-7$ i.e. $m\ge\frac{7+\sqrt{217}}{12}$.

Now using the condition $1/p + 1/q = 1/2$ from section 3.6 and the expression $p = \frac{1}{3}(5m-2)$ for p, we obtain $q = \dfrac{2p}{p-2} = \dfrac{10m-4}{5m-8}$.

To be compatible with inequality (6.1), we need $0 \leq 5m^2 + 3m - 20$ i.e. $m \geq \frac{-3+\sqrt{409}}{10}$. Now taking the maximum of the three lower bounds, we have that the theorem is valid for $m^* = \frac{7+\sqrt{217}}{12}$.

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E-mail address: difrance@univaq.it
E-mail address: a.lorz@damtp.cam.ac.uk
E-mail address: p.a.markowich@damtp.cam.ac.uk